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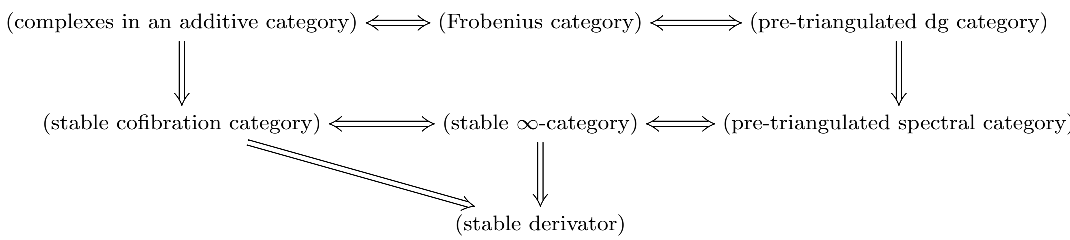
WS 2019/20

Graduate Seminar Topology (S4D2)
'Enhancements of triangulated categories'

Di 14:15–15:45, SR N 0.008 (Annex)

Triangulated categories are a useful language in several areas of pure mathematics, such as algebra, representation theory, algebraic geometry and algebraic topology. Ever since triangulated categories were introduced by Puppe and Verdier, there was a consensus that they are a truncation of certain 'higher structures'. The higher structure can be made rigorous in various different ways; the aim of this seminar is to explain how to 'enhance' triangulated categories, and to compare the enhancements. The following diagram gives an overview about the different concepts and their interrelations:



Talk 1 (08.10.19): Triangulated categories
Definition of a triangulated category [Ve]; Neeman's formulation of the octahedral axiom [Ne]; Example: homotopy category of chain complexes in an additive category [Ve, II.1.3.2], [W, Prop. 10.2.4].

Talk 2 (15.10.19): Stable and derived categories
Example: stable category of a Frobenius category [Ke06, 3.3], [Ha, Thm. 9.4]; comparison with the homotopy category of an additive category; example: derived category of an abelian category (with variations such as boundedness, etc...).

Talk 3 (29.10.19): Algebraic triangulated categories
Example: homology category of a pre-triangulated dg category [BK, §3], [Ke06, 2.2], [Sch13a, Sec. 2]; equivalence of the three characterizations of algebraic triangulated categories [Kr, 7.5], [Ke06].

Talk 4 (05.11.19): Derivators
Definition of derivators and stable derivators [Gr13, Gr]; construction of shifts and distinguished triangles, proof of the triangulation [Gr, Part III]; example: extend algebraic triangulated categories to a derivator. (There are variations on the concept of a derivator, such as the 'homotopy theories' of Heller [He88, He97], the 'epivalent towers' of Keller [Ke91] and the 'systems of triangulated diagram categories' of Franke [Fra, Sec. 1]; the talk should concentrate on derivators, though).

Talk 5 (12.11.19): Stable ∞-categories
Definition of ∞-categories (i.e., quasi-categories) and stability [Lu09, Ch. 1], [LuHA, Ch. 1], [Gr10, Ch. 1, 2, 5]; definition of the homotopy category and its triangulation (in the stable case); the derivator of a stable ∞-category.

Talk 6 (19.11.19): Stable cofibration categories
Definition of a cofibration category [Br, I.1], [Sch13t, Def. 1.1], [R-B]; definition of the homotopy category and its triangulation (in the stable case) [Sch13t, App.]; stable model categories [Q, DS, Ho99, SS03] as a special case; examples of stable cofibration categories [Sch13t, Sec. 1], [SS03, 2.3, 2.4].

Talk 7 (26.11.19): ∞-categories versus cofibration categories
The ∞-category of a cofibration category [Sz14, Ch. 3]; the cofibration category of an ∞-category [Sz14, Ch. 4]; Explain how the constructions are an equivalence of homotopy theories [Sz14, Thm. 4.11] and show that the notions of 'stability' correspond. The results of [Sz14] were published as the papers [Sz1, Sz2, Sz3], so you may also want to consult these references.

Talk 8 (03.12.19): Spectral categories
Definition of a spectral category and its homotopy category [SS03, Def. 3.3.1] [BM, Sec. 3]; triangulation for pre-triangulated spectral categories [BM, Thm. 4.6]; comparison with stable model categories [SS03]; comparison with dg categories [BM, Def. 2.9].

Talk 9 (10.12.19): Universal properties of stable homotopy theory
Spectra and the stable homotopy category [BF]; universal property in the context of stable model categories [SS02], and the action on topological triangulated categories [Le]; universal property in the context of ∞-categories; universal property in the context of derivators [Fra] and the action on stable derivators.

Talk 10 (17.12.19): Topological versus algebraic triangulated categories
Definition of topological triangulated categories [Sch13t, Def. 1.4]; algebraic triangulated categories are topological [Sch13t, Ex. 1.6]; obstructions to algebraizicity [Sch10] [Sch13a, Thm. 2.1]; the p-local stable homotopy category is not algebraic [Sch13a, Thm. 1.3].

Talk 11 (14.01.20): Exotic triangulated categories
[MSS]

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