MORITA EQUIVALENCE IN ALGEBRA AND GEOMETRY

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Abstract. We study the notion of Morita equivalence in various categories. We start with Morita equivalence and Morita duality in pure algebra. Then we consider strong Morita equivalence for C*-algebras and Morita equivalence for W*-algebras. Finally, we look at the corresponding notions for groupoids (with structure) and Poisson manifolds.

1. Algebraic Morita Equivalence

The main idea of Morita equivalence in pure algebra can be illustrated by the following example. Let R be any ring with unit, let $\mathbb{M}_n(R)$ be the ring of $n \times n$-matrices over R for some $n \in \mathbb{N}$. If V is a (left) R-module, then V^n is a $\mathbb{M}_n(R)$-module in a canonical way (matrix-vector multiplication), and the correspondence $V \mapsto V^n$ is functorial. Conversely, every $\mathbb{M}_n(R)$-module can be so obtained from some R-module. Thus the rings R and $\mathbb{M}_n(R)$ have equivalent categories of left modules.

Definition 1. We write ${}_R\mathfrak{M}$ for the category of left R-modules. Two unital rings are called *Morita equivalent* if they have equivalent categories of left modules.

There is also a useful theory of Morita equivalence for rings with a "set of local units", i.e. sufficiently many idempotents (cf. [1]), but things become far more complicated. Unless we have useful topologies around, as in the case of C*-algebras, we assume all our rings to be unital.

Let R and S be rings with unit. There is a standard way to get a functor from ${}_R\mathfrak{M}$ to ${}_S\mathfrak{M}$: If ${}_SQ_R$ is any (S,R)-bimodule and V is an R-module, then ${}_SQ_R \otimes_R V$ carries a natural S-module structure. Thus every (S,R)-bimodule induces a functor from ${}_R\mathfrak{M}$ to ${}_S\mathfrak{M}$. Taking the tensor product of bimodules corresponds to the composition of these functors. Conversely, under some hypotheses, every (covariant) functor must be of this form:

Theorem 1 (Watts [21]). *Let T be a right-exact covariant functor from ${}_R\mathfrak{M}$ to ${}_S\mathfrak{M}$ which commutes with direct sums. Then there is an (S,R)-bimodule Q such that the functors T and $Q \otimes_R \sqcup$ are naturally equivalent. Moreover, Q is unique up to isomorphism of bimodules.*

This result was discovered simultaneously by Eilenberg, Gabriel, and Watts around 1960. As usual in homological algebra, the proof is trivial. Notice that every equivalence of categories has to preserve direct sums and exact sequences and thus satisfies the hypotheses of Theorem 1. Hence we obtain

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Corollary 1.1. *Two rings R and S are Morita equivalent if and only if there are bimodules ${}_RP_S$ and ${}_SQ_R$ such that ${}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R$ and ${}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S$ as bimodules.*

This result implies that Morita equivalent rings also have equivalent categories of right modules and bimodules. It is also easy to see that they have equivalent lattices of ideals, so that the properties of being Noetherian, Artinian, or simple are Morita invariant (cf. [4]). They have isomorphic categories of projective modules and thus equivalent K-theories. More generally, a decent (co)homology theory should be Morita invariant, and this is indeed true for cyclic homology, Hochschild homology (for k-algebras) (cf. [9]).

Moreover, Morita equivalent rings have isomorphic centers. This implies that Morita equivalent Abelian rings are already isomorphic. Thus Morita equivalence is essentially a non-commutative phenomenon. This gives another reason why so many homology functors are Morita invariant: Usually, they arise as extensions of functors defined on a category of commutative algebras to a category of non-commutative algebras. But Morita invariance imposes no restrictions whatsoever on functors defined on a category of commutative algebras, so that we can hope for a Morita invariant extension. Examples show that if a functor can be extended "naturally", then the extension tends to be indeed Morita invariant.

An important problem is to find conditions when two rings are equivalent. Notice that we do not have to find two bimodules P and Q because one of them determines the other. In general, if the bimodules P and Q implement a Morita equivalence between R and S, we have

$$Q \cong \mathrm{Hom}_S(P, S) \cong \mathrm{Hom}_R(P, R), \qquad P \cong \mathrm{Hom}_S(Q, S) \cong \mathrm{Hom}_R(Q, R).$$

This means that Q and P are in some sense dual to each other. In the purely algebraic setting, there is no natural way to turn an (S, R)-bimodule into an (R, S)-bimodule; the nearest we can get is the above relation between P and Q. For C*-algebras or groupoids, we *can* turn left actions into right actions using the adjoint operation or inversion, which will slightly simplify matters there.

We let $\mathrm{End}(Q)$ be the ring of endomorphisms of the additive group Q, i.e. Q without the (S, R)-bimodule structure. Then the right/left operations of R and S on Q induce injective homomorphisms $R \to \mathrm{End}(Q)$ and $S \to \mathrm{End}(Q)$. The bimodule property asserts that the images of R and S in $\mathrm{End}(Q)$ commute, and in order to have a Morita equivalence, they must be the full commutants of each other, i.e. $R' = S$, $S' = R$. This is clear because $Q \otimes_R V$ is an R'-module for all $V \in {}_R\mathfrak{M}$, and if our inducing process gives all S-modules, we need $R' = S$. Thus R and the right module structure of Q determine S as the commutant of R in $\mathrm{End}(Q)$. Of course, not every module Q_R induces a Morita equivalence (e.g. the zero module does not work).

Theorem 2 (Morita [11], [12])**.** *Let R be a ring with unit and Q a right R-module. Then Q induces a Morita equivalence between R and $R' \subset \mathrm{End}(Q)$ if and only if Q_R is a finitely generated projective generator.*[1]

Of course, the idea of "representation equivalence" is older than Morita's work. His main contribution was to make formal definitions and to put the various uses of

[1] An object X in an Abelian category is a generator iff every object is a quotient of a direct sum of copies of X.

this idea into a general theory. Besides the notion of equivalence, Morita also studied a corresponding duality. Formally, this consists of replacing covariant functors by contravariant functors. Since we have $V^{**} \neq V$ for an infinite-dimensional vector space, duality can only hold if one restricts attention to finitely generated modules and assumes that the underlying ring is Noetherian. Under these assumptions, the theory goes through smoothly and yields:

Definition 2 (Morita [11])**.** Let ${}_R\mathfrak{F}$ be the category of finitely generated left R-modules. If R and S are unital Noetherian rings, a duality is a pair $T\colon {}_R\mathfrak{F} \to {}_S\mathfrak{F}$, $U\colon {}_S\mathfrak{F} \to {}_R\mathfrak{F}$ of contravariant equivalence functors.

Theorem 3 (Morita [11])**.** *Let R and S be Noetherian rings. If there is a duality (T, U) between ${}_R\mathfrak{F}$ and ${}_S\mathfrak{F}$, then there exists a bimodule ${}_SQ_R$ such that $T \cong \mathrm{Hom}_R(\sqcup, Q)$, $U \cong \mathrm{Hom}_S(\sqcup, Q)$. Moreover, the maps $R, S \to \mathrm{End}(Q)$ are injective, and $R' = S$, $S' = R$.*

Morita also has a necessary and sufficient condition for Q to induce a Morita duality.

2. Morita equivalence for C*-algebras and W*-algebras

In these categories, we have considerably more structure and therefore restrict our categories of modules.

Definition 3 (Rieffel [16])**.** A *Hermitian module* over a C*-algebra A is the Hilbert space $\mathfrak{H}$ of a non-degenerate $*$-representation $\pi\colon A \to \mathrm{B}(\mathfrak{H})$, together with the action $a \cdot \xi = \pi(a)\xi$ for $a \in A$, $\xi \in \mathfrak{H}$. If A is even a W*-algebra, we assume in addition that π is a normal map[2]and call $\mathfrak{H}$ a *normal A-module*. In both cases, morphisms are the intertwining operators, i.e. A-module homomorphisms in the usual algebraic sense.

We call two C*-algebras *Morita equivalent* if they have equivalent categories of Hermitian modules and if the equivalence functors T are $*$-functors, i.e. if $f\colon V_1 \to V_2$ is a morphism, then $T(f^*) = (Tf)^*$. Similarly, we call two W*-algebras *Morita equivalent* if they have equivalent categories of normal modules and if the equivalence is implemented by $*$-functors.

The category of Hermitian modules over a C*-algebra A is equivalent to the category of normal modules over the enveloping von Neumann algebra $n(A)$. Hence Morita equivalence of C*-algebras is really a von Neumann algebra concept and too weak for most applications. We will soon define the more restrictive concept of strong Morita equivalence for C*-algebras. As in the purely algebraic case, we need more concrete criteria in terms of bimodules for two algebras to be equivalent. Since we have to transport the Hilbert space inner products, we need to put more structure on our bimodules:

Definition 4 (Paschke [14], Rieffel [15])**.** Let B be a C*-algebra. A *pre-Hilbert B-module* is a right B-module X (with a compatible $\mathbb{C}$-vector space structure), equipped with a conjugate-bilinear map (linear in the second variable) $\langle \sqcup, \sqcup \rangle_B : X \times X \to B$ satisfying

[2] In a W*-algebra, every bounded increasing net of positive elements has a least upper bound. A positive map $f\colon A \to B$ between W*-algebras is called *normal* if, for any bounded increasing net (p_j) of positive elements of A with least upper bound p_∞, $f(p_\infty)$ is the least upper bound of the net $(f(p_j))$.

1. $\langle x, x\rangle_B \geq 0$ for all $x \in X$;
2. $\langle x, x\rangle_B = 0$ only if $x = 0$;
3. $\langle x, y\rangle_B = \langle y, x\rangle_B^*$ for all $x, y \in X$;
4. $\langle x, y \cdot b\rangle_B = \langle x, y\rangle_B \cdot b$ for all $x, y \in X$, $b \in B$.

The map $\langle \sqcup, \sqcup\rangle_B$ is called a *B-valued inner product on X*.

It can be shown that $\|x\| = \|\langle x, x\rangle_B\|^{1/2}$ defines a norm on X. If X is complete with respect to this norm, it is called a *Hilbert B-module*. If not, all the structure can be extended to its completion to turn it into a Hilbert B-module. Actually, in Paschke's paper, the inner product is linear in the first variable; and in Rieffel's paper, this object is called a right (pre-)B-rigged space.

This contains enough structure to transport Hilbert space inner products: If V is a Hermitian B-module and X is a Hilbert B-module, we can equip the algebraic tensor product $X \otimes_B V$ with an inner product

$$\langle x \otimes v, x' \otimes v'\rangle = \langle\langle x', x\rangle_B v, v'\rangle_V,$$

where $\langle \sqcup, \sqcup\rangle_V$ is the inner product on V. It can be shown that this is non-negative definite and thus defines a pre-inner product on $X \otimes_B V$. Thus factoring out the vectors of lenth zero and completing gives a new Hilbert space $X \mathbin{\overline{\otimes}}_B V$. This construction is functorial: If $f\colon V_1 \to V_2$ is a morphism of Hermitian B-modules, then $\mathrm{id} \otimes f$ extends to a bounded map $X \mathbin{\overline{\otimes}}_B V_1 \to X \mathbin{\overline{\otimes}}_B V_2$.

If $e \in \mathrm{lin}(X, X)$ is a bounded operator commuting with the action of B by right multiplication, then $e \otimes \mathrm{id}_V$ extends to a bounded operator on $X \mathbin{\overline{\otimes}}_B V$. However, the commutant of B, in general, is not a C*-algebra because bounded operators may fail to have an *adjoint*. If $T \in \mathrm{lin}(X, X)$, an operator $T^* \in \mathrm{lin}(X, X)$ is called an adjoint for T if $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in X$. Let E be the algebra of all *adjointable* operators on X, i.e. operators that have an adjoint. It is easy to see that an adjointable operator is necessarily bounded and commutes with the action of B. Moreover, E with the natural norm becomes a C*-algebra. It is easy to see that $X \mathbin{\overline{\otimes}}_B V$ is a Hermitian E-module as expected. Moreover, the mappings $X \mathbin{\overline{\otimes}}_B V_1 \to X \mathbin{\overline{\otimes}}_B V_2$ induced by B-module homomorphisms are E-module homomorphisms as desired, so that X induces a functor from B- to E-modules.

Let $B_0 \subset B$ be the closed linear span of $\langle X, X\rangle_B = \{\langle x, y\rangle, x, y \in X\}$. If B_0 acts trivially on V, then $X \mathbin{\overline{\otimes}}_B V$ is the zero module, so that the functor induced by X fails to be faithful. Similarly, the algebra E may be too big.

This can easily be seen from the example $B = \mathbb{C}$, $X = \mathfrak{H}$ infinite-dimensional. In this case, $E = \mathrm{B}(\mathfrak{H})$ is the algebra of all bounded operators on $\mathfrak{H}$. But $\mathrm{B}(\mathfrak{H})$ has the non-trivial ideal $\mathrm{K}(\mathfrak{H})$ of compact operators. It is well-known that $\mathrm{K}(\mathfrak{H})$ is Morita equivalent to $\mathbb{C}$: This means that every irreducible representation of $\mathrm{K}(\mathfrak{H})$ is a (possibly infinite) direct sum of copies of the standard representation. But $\mathrm{B}(\mathfrak{H})$, as a C*-algebra has more complicated representations coming from the Calkin algebra $\mathrm{B}(\mathfrak{H})/\mathrm{K}(\mathfrak{H})$. Here we have to be careful: As a W*-algebra, $\mathrm{B}(\mathfrak{H})$ is Morita equivalent to $\mathbb{C}$, but not if we view it as a C*-algebra.

This example suggests to look for an analogue of the ideal of compact operators for Hilbert modules. The right approach is to let E_0 be the closed linear span of the "rank one operators" $\langle x, y\rangle_E \in E$ given by $\langle x, y\rangle_E z = x\langle y, z\rangle_B$ for $x, y, z \in X$. It is easily seen that E_0 is an ideal in E. Moreover, now the roles of E_0 and B_0 are symmetric: We have just defined an E_0-valued inner product on X and X is an E_0-module by definition, only that we have exchanged left and right.

Definition 5 (Rieffel [15], [17])**.** Let E and B be C^*-algebras. By an *E-B-equivalence bimodule* we mean an E, B-bimodule which is equipped with E- and B-valued inner products with respect to which X is a right Hilbert B-module and a left Hilbert E-module such that

1. $\langle x, y\rangle_E z = x\langle y, z\rangle_B$ for all $x, y, z \in X$;
2. $\langle X, X\rangle_B$ spans a dense subset of B and $\langle X, X\rangle_E$ spans a dense subset of E.

We call E and B *strongly Morita equivalent* if there is an E-B-equivalence bimodule.

If X is an E-B-equivalence bimodule, it is easy to endow the conjugate space $\tilde{X}$, which is X as a set with the same addition and scalar multiplication $\lambda\tilde{x} = (\overline{\lambda}x)\tilde{}$, with the structure of a B-E-equivalence bimodule. For example, $\tilde{x}e = (e^*x)\tilde{}$. Moreover, it is not difficult to see that strong Morita equivalence is an equivalence relation.

Theorem 4 (Rieffel [15])**.** *Let X be an E-B-equivalence bimodule. Then $X \otimes_B \sqcup$ induces an equivalence between the category of Hermitian B-modules and the category of Hermitian E-modules, the inverse being given by $\tilde{X} \otimes_E \sqcup$. This functor preserves weak containment and direct integrals.*

The reason for Rieffel to introduce strong Morita equivalence was to improve the understanding of induced representations of (locally compact) groups. Let G be a l.c. group and let H be a closed subgroup. Then unitary representations of H "induce" representations of G. Moreover, the representations of G obtained by this process are precisely those that admit a "system of imprimitivity". In more modern language, the representations that can be obtained by inducing from H are the covariant representations of $(\mathrm{C}_\infty(G/H), G)$, where $\mathrm{C}_\infty(G/H)$ are the functions on G/H vanishing at infinity and the action of G on $\mathrm{C}_\infty(G/H)$ is obtained from the left translation action of G on G/H. These results are due to Mackey (for the separable case), but his proofs were based on rather unintuitive measure theoretic arguments. In [15], Rieffel gave a new proof by showing that the group algebra $C^*(H)$ is strongly Morita equivalent to the crossed product $\mathrm{C}_\infty(G/H) \rtimes G$. Actually, he worked with the dense subalgebras $\mathrm{C}_c(\sqcup)$ of functions of compact support and showed that $\mathrm{C}_c(G)$ can be given the structure of a pre-Hilbert $\mathrm{C}_c(H)$-module. Then he identified the algebra of "finite rank operators" on $\mathrm{C}_c(G)$ with a dense subalgebra of the crossed product $\mathrm{C}_\infty(G/H) \rtimes G$.

But there are also other, more non-commutative applications. For example, if G is a compact group acting on a C^*-algebra A by automorphisms, $\alpha\colon G \to \mathrm{Aut}(A)$, we can define a "conditional expectation" $p\colon A \to A^\alpha$, where A^α is the fixed point algebra, by averaging $p(a) = \int_G \alpha_x(a)\,dx$ with respect to Haar measure dx. Then $\langle a, b\rangle_{A^\alpha} = p(a^*b)$ turns A into a pre-Hilbert A^α-module. It can be shown that this gives us a strong Morita equivalence of A^α with a certain ideal of the crossed product algebra $A \rtimes_\alpha G$. In the commutative case, this ideal is the whole crossed product algebra, if and only if the action of G is free. Actually, the case where G is not compact is very important but also much more subtle (cf. [19]).

Another more elementary example is the following: Let $p \in A$, then the corresponding left ideal Ap can be made into an $\overline{ApA}$-$\overline{pAp}$-equivalence bimodule with inner products $\langle x, y\rangle_{ApA} = xy^*$, $\langle x, y\rangle_{pAp} = x^*y$. Subalgebras of the form $\overline{pAp}$ are the prototype of hereditary subalgebras, and the corresponding hereditary subalgebra is called full if $\overline{ApA} = A$. This example is of considerable theoretical importance because every strong Morita equivalence is of this form: If A and B are strongly Morita equivalent C^*-algebras, there is a C^*-algebra C that contains both A and B as full hereditary subalgebras. Together with a result of Brown on hereditary subalgebras in [2], this gives the following remarkable theorem:

Theorem 5 (Brown-Green-Rieffel [3]). *Let A and B be C^*-algebras with a countable approximate identity (e.g. separable or unital). Then they are strongly Morita equivalent if and only if they are stably equivalent, i.e. $A \otimes \mathrm{K} \cong B \otimes \mathrm{K}$, where K is the algebra of compact operators on a separable Hilbert space.*

Thus stable equivalence, which is of considerable importance in K-theory, can be viewed as a separable version of Morita equivalence. Moreover, since the class of separable or unital algebras is already rather large, one can expect that properties that are invariant under stable equivalence are also Morita invariant. For example, Morita equivalent C*-algebras have isomorphic lattices of ideals and the same K-, E-, and KK-theory. In [6], it is shown how to induce traces between Morita equivalent C*-algebras. In [5], Morita equivalence for group actions on C*-algebras is defined, and it is shown that equivalent group actions give rise to Morita equivalent group C*-algebras and reduced group C*-algebras.

Now let us briefly discuss the situation for von Neumann algebras. If M is a von Neumann algebra, a further requirement for ("normal") Hilbert M-N-bimodules is that the maps $m \mapsto \langle x, my\rangle_N$ be σ-weakly continous for all $x, y \in X$. On the other hand, we can weaken the requirements for an equivalence bimodule, replacing density by weak density. That this is possible is illustrated by the example $\mathbb{C}$, $\mathrm{B}(\mathfrak{H})$. With these changes, the analogue of the Eilenberg-Gabriel-Watts theorem is again true:

Theorem 6 (Rieffel [16]). *Let M and N be W^*-algebras. Then every normal equivalence bimodule implements an equivalence between the categories of normal M- and N-modules by a $*$-functor. Conversely, every such equivalence is implemented by some normal equivalence bimodule.*

It is easy to see that Morita equivalent von Neumann algebras have isomorphic centers and isomorphic lattices of weakly closed ideals. Moreover, if M and N are Morita equivalent and if M is of type $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, then the same holds for N, i.e. Morita equivalence respects the type of a von Neumann algebra. For types I and III, the classification up to Morita equivalence is very easy:

Theorem 7 (Rieffel [16]). *Two W^*-algebras of type I are Morita equivalent if and only if they have isomorphic centers. Two von Neumann algebras of type III on separable Hilbert spaces are Morita equivalent if and only if they are isomorphic.*

As pointed out to me by Dimitri Shlyakhtenko, two factors M, N of type II on separable Hilbert spaces are equivalent if and only if they are stably equivalent as von Neumann algebras, i.e. $M \otimes \mathrm{B}(\mathfrak{H}) \cong N \otimes \mathrm{B}(\mathfrak{H})$ (this tensor product is in the category of von Neumann algebras and is defined to be the weak closure of the spatial tensor product). Thus every type II_1-factor is equivalent to a II_∞-factor, and conversely. The idea of the proof is to turn a M-N-Hilbert bimodule for two II_1-factors into a genuine (pre-)Hilbert space using the trace on one of them. The actions extend to the completion, and it turns out that M and N are commutants of one another. Hence we obtain a correspondence in the sense of [20] and can apply the theory for those.

It should be remarked that Morita equivalence of von Neumann algebras is not an important technical tool, but at most a convenient way of formulating some of the known results. For example, a von Neumann algebra is of type I iff it is Morita equivalent to a commutative von Neumann algebra.

3. Morita equivalence for topological and symplectic groupoids

Now we look at geometric analogues of Morita equivalence, first for locally compact topological groupoids. The bimodule version still makes sense:

Definition 6 (Muhly-Renault-Williams [13]). Let G be a locally compact topological groupoid with unit space $G^{(0)}$ and source and range maps s and r. A locally compact space X with a continuous, open map $\rho\colon X \to G^{(0)}$, which we call the *momentum map* and an action $\mu\colon G * X \to X$, where $G * X = \{(g,x) \in G \times X \mid s(g) = \rho(x)\}$, is called a *left G-space* if

1. $\rho\bigl(\mu(g,x)\bigr) = r(g)$ for all $(g,x) \in G * X$;
2. $\mu\bigl(\epsilon(\rho(x)), x\bigr) = x$ for all $x \in X$; and
3. $\mu(g \cdot h, x) = \mu\bigl(g, \mu(h,x)\bigr)$ whenever $(g,h) \in G * G$ and $(h,x) \in G * X$.

We write $g \cdot x = gx = \mu(g,x)$. A *right G-space* is defined similarly.

The action is called *free* if $(g,x) \in G * X$ and $g \cdot x = x$ implies $g \in G^{(0)}$, i.e. only units have fixed points.

The action is called *proper* if the map $(\mu, \mathrm{id})\colon G * X \to X \times X$ sending (g,x) to $(g \cdot x, x)$ is proper.

If H is another groupoid and if X is at the same time a left G-space and a right H-space with momentum maps $\rho\colon X \to G^{(0)}$ and $\sigma\colon X \to H^{(0)}$, we call it a *$G$-$H$-bimodule* if the actions commute, i.e.

1. $\rho(x \cdot h) = \rho(x)$ for all $(x,h) \in X * H$ and similarly $\sigma(g \cdot x) = \sigma(x)$ for all (g,x) in $G * X$; and
2. $g \cdot (x \cdot h) = (g \cdot x) \cdot h)$ for all $(g,x) \in G * X$, $(x,h) \in X * H$.

We say that a G-H-bimodule X is an *equivalence bimodule* if

1. it is free and proper both as a G- and an H-space;
2. the momentum map $\rho\colon X \to G^{(0)}$ induces a bijection of X/H to $G^{(0)}$; and
3. the momentum map $\sigma\colon X \to H^{(0)}$ induces a bijection of $G\backslash X$ to $H^{(0)}$.

We call G and H *Morita equivalent* if a G-H-equivalence bimodule exists.

The orbit space for a proper groupoid action is always locally compact Hausdorff, and the projection onto the orbit space is open. Thus for an equivalence bimodule the bijections $X/H \cong G^{(0)}$, $G\backslash X \cong H^{(0)}$ are automatically homeomorphisms.

The action of a groupoid on itself by left and right multiplication turns it into a G-G-equivalence bimodule, so that Morita equivalence is a reflexive relation. It is easy to see that it is also symmetric and transitive. For the latter one uses the analogue $X *_H Y$ of the bimodule tensor product: If X and Y are G-H- and H-K-bimodules respectively, then $X * Y = \{(x,y) \in X \times Y \mid \sigma_X(x) = \rho_Y(y)\}$, where $\sigma_X : X \to H^{(0)}$ and $\rho_Y : Y \to H^{(0)}$ are the momentum maps. In order to get $X *_H Y$, identify $(x \cdot h, y) \sim (x, h \cdot y)$ when this is defined. It is not difficult to endow this with the structure of a locally compact G-K-space and to see that this process produces equivalence bimodules if X and Y were equivalence bimodules. Moreover, this tensor product is functorial (for "equivariant" continuous maps as morphisms).

Corollary 7.1. *Let G and H be Morita equivalent locally compact groupoids. Then the categories of left (right) G- and H-spaces are equivalent.*

As in the algebraic case, under suitable hypotheses a left G-space determines a groupoid H such that it becomes a G-H-equivalence bimodule [13]. To be more specific, let X be a free proper G-space with a surjective momentum map ρ. Let $X * X = \{(x, y) \in X \times X \mid \rho(x) = \rho(y)\}$. Then G acts freely and properly on $X * X$ by the diagonal action $g(x, y) = (gx, gy)$. The orbit space $H = G\backslash X * X$ can be endowed naturally with a groupoid structure over $G\backslash X$ by putting $[x, y] \cdot [y, z] = [x, z]$, and this multiplication is continuous. There is an obvious right action of H on X defined by $x \cdot [x, y] = y$. It can be checked that this turns X into a G-H-equivalence bimodule. Moreover, if X was a G-H'-equivalence bimodule to start with, then we get $H \cong H'$.

There are many examples of Morita equivalent groupoids [13]. If G is a transitive groupoid, $u \in G^{(0)}$, then $r^{-1}(u)$ is an equivalence bimodule for G and the isotropy group $r^{-1}(u) \cap s^{-1}(u)$ at u, if r and s are open maps. A similar statement holds if $U \subset G^{(0)}$ is a subset meeting every G-orbit. This applies especially to foliations (transverse submanifold meeting every leaf). Moreover, we get that the groupoid associated to a (Cartan) principal bundle (cf. [8]) is equivalent to the structure group of the bundle.

Another typical example is the following situation: Let H and K be locally compact groups acting freely and properly on a locally compact Hausdorff space P such that the actions commute. Let H act on the left and K act on the right. The commutativity assumption means that we get an action of K on $H\backslash P$ and an action of H on P/K. Then the space P is an equivalence for the transformation groupoids $(H, P/K)$ and $(K, H\backslash P)$.

How is Morita equivalence of groupoids related to the algebraic notion? Fix Haar systems λ and β for G and H. Then we can form the (full) groupoid C*-algebras $C^*(G, \lambda)$ and $C^*(H, \beta)$ with respect to these Haar systems. For the groupoids coming from the last example above, it was already discovered by Green (cf. [18]) that the associated groupoid C*-algebras are Morita equivalent. In [13], it is shown that this remains true in general, with a proof similar to Rieffel's argument in [18]:

Theorem 8 (Muhly-Renault-Williams [13])**.** *Let G and H be locally compact, second countable, Hausdorff groupoids with Haar systems λ and β. If there is a (G, H)-equivalence bimodule X, then the (full) groupoid C*-algebras $C^*(G, \lambda)$ and $C^*(H, \beta)$ are strongly Morita equivalent.*

The definition of the groupoid C*-algebra depends on the choice of a Haar system. However, the definition of a representation of a groupoid does not. In the group case, Haar measure is essentially unique, but for groupoids, this is no longer the case. Due to the correspondence of groupoid representations and representations of the groupoid C*-algebra, different choices of Haar system certainly produce Morita equivalent C*-algebras. This still leaves open whether we actually get isomorphic C*-algebras. At least in the case of transitive groupoids, this is indeed the case:

Theorem 9 (Muhly-Renault-Williams [13])**.** *Let G be a second countable, locally compact, transitive groupoid, let $u \in G^{(0)}$, and let H be the isotropy group at u. Let λ be a Haar system for G. Then there is a positive measure μ on $G^{(0)}$ of full support such that $C^*(G, \lambda)$ is isomorphic to $C^*(H) \otimes \mathrm{K}\big(L^2(G^{(0)}, \mu)\big)$.*

It is easy to see that $C^*(G, \lambda)$ must be strongly Morita equivalent to $C^*(H)$. But the above refinement shows that we do not have to tensor $C^*(G, \lambda)$ with the compact operators. This shows that the groupoid algebra is stable and does not depend on the choice of Haar system.

By the way, it probably is not very interesting to look for criteria on groupoids that are necessary and sufficient for the groupoid C*-algebras to be Morita equivalent. This can already be seen by looking at groups. It is easy to see that two groups are equivalent in the sense of Definition 6 iff they are isomorphic as topological groups. However, if K is a compact group, then by the Peter-Weyl theorem its groupoid C*-algebra is a direct sum of copies of full matrix algebras, and there are infinitely many such copies if and only if K is has infinitely many elements. Thus any two infinite compact groups have strongly Morita equivalent, even stably equivalent, group C*-algebras. However, there seems to be no natural equivalence bimodule in this situation that can be written down without knowing the full representation theory of the involved groups.

If we drop all continuity assumptions, we get a notion of Morita equivalence for algebraic groupoids without any further structure. More importantly, if our groupoids carry additional differentiability structure, we should strengthen our requirements on equivalences by asserting that the actions are smooth in order to get an equivalence of the categories of smooth actions. Moreover, the bijections of the orbit spaces X/H with $G^{(0)}$ and $G\backslash X$ with $H^{(0)}$ should be diffeomorphic. This follows if the momentum maps are *full*, i.e. surjective submersions.

4. Morita equivalence for symplectic groupoids and Poisson manifolds

Definition 7 (Xu [23], [25])**.** Two symplectic groupoids G and H with unit spaces $G^{(0)}$ and $H^{(0)}$ are called *Morita equivalent* if there are a symplectic manifold X and surjective submersions $\rho\colon X \to G^{(0)}$ and $\sigma\colon X \to H^{(0)}$ such that

1. G has a free, proper, symplectic [10] left action on X with momentum map ρ;
2. H has a free, proper, symplectic right action on X with momentum map σ;
3. the two actions commute with each other;
4. ρ induces a diffeomorphism $X/H \to G^{(0)}$;
5. σ induces a diffeomorphism $G\backslash X \to H^{(0)}$;

$(X;\rho;\sigma)$ is called an *equivalence bimodule between G and H*.

As expected, Morita equivalence is an equivalence relation among symplectic groupoids. Since the notion is stronger than equivalence for topological groupoids, Morita equivalent symplectic groupoids still have Morita equivalent groupoid C*-algebras. Equivalence bimodules for symplectic groupoids were also studied from a slightly different viewpoint and under the name of an affinoid structure by Weinstein in [22].

The point of introducing the stronger relation above is that we now get results about the category of symplectic actions of our groupoids that are completely analogous to the results for locally compact groupoids. In the proofs, it only has to be checked that the symplectic structure can be transported. On the other hand, topological problems almost disappear in this category. It is easy to see that Morita equivalence of symplectic groupoids is an equivalence relation.

Theorem 10 (Xu [23], [25])**.** *Let G be a symplectic groupoid over $G^{(0)}$ and let $\rho\colon X \to G^{(0)}$ be a full, symplectic, free, and proper left G-module. Then $G\backslash X$ is a Poisson manifold, and $H = G\backslash(X^- *_G X)$ is a symplectic groupoid over $G\backslash X^-$*

in a natural way. Moreover, $\sigma\colon X \to G\backslash X^-$ *naturally becomes a symplectic right* H*-module such that* $(X;\rho;\sigma)$ *is an equivalence bimodule between* G *and* H*.*

Conversely, if $(X;\rho;\sigma)$ *is any equivalence bimodule between symplectic groupoids* G *and* H*, then* $H \cong G\backslash(X^- *_G X)$ *as symplectic groupoids.*

As usual, if P is a Poisson manifold with bracket $[,]$, then P^- denotes the same manifold with bracket $-[,]$.

Let G be a symplectic groupoid. We can consider the "category" of symplectic left modules over G, in which morphisms between symplectic modules F_1 and F_2 are Lagrangian submanifolds of $F_1 *_G F_2$ invariant under the diagonal action of G, and the composition of morphisms is the set theoretic composition of relations. (This is not a true category since the composition of two morphisms need not be a submanifold.) Then we obtain

Theorem 11 (Xu [23], [25])**.** *Morita equivalent symplectic groupoids have equivalent "categories" of symplectic left modules.*

One motivation for introducing Morita equivalence of symplectic groupoids is the correspondence between integrable Poisson manifolds and r-simply connected[3] symplectic groupoids. It is possible to pull back the groupoid equivalence to the base Poisson manifolds:

Definition 8 (Xu [23], [24])**.** Two Poisson manifolds P_1 and P_2 are *Morita equivalent* if there exists a symplectic manifold X together with complete Poisson morphisms $\rho\colon X \to P_1$ and $\sigma\colon X \to P_2^-$ that form a full dual pair with connected and simply connected fibers. Then X is called an equivalence bimodule.

The reason for requiring connected simply connected fibers is to exclude certain cases that we do not want to be Morita equivalences. For example, with this definition two connected symplectic manifolds are Morita equivalent iff they have the same fundamental group. This somewhat complicated definition is borne to make true the following theorem:

Theorem 12 (Xu [23] [24])**.** *Let* P_1 *and* P_2 *be integrable Poisson manifolds. Then* P_1 *and* P_2 *are Morita equivalent if and only if their* r*-simply connected symplectic groupoids are Morita equivalent.*

Let G be an r-simply connected groupoid over P. The main step in the proof of Theorem 12 is to show that if X is a symplectic left G-module, then its momentum map $\rho\colon X \to P$ is a complete symplectic realization and that, conversely, every complete symplectic realization of P carries a natural left G-action. This also proves the following theorem:

Theorem 13 (Xu [23], [24])**.** *Equivalent integrable Poisson manifolds have equivalent "categories" of complete symplectic realizations.*

Furthermore, Theorem 12 immediately implies that Morita equivalence is an equivalence relation among integrable Poisson manifolds. This is not true for arbitrary Poisson manifolds: Already reflexivity fails. Currently, it is not even known whether every Poisson manifold has a *complete* symplectic realization.

It is not difficult to see that an equivalence bimodule between two Poisson manifolds induces a bijection between their leaf spaces. Moreover, Morita equivalence

[3] A groupoid is called r- ... if all r-fibers have the property .... Many authors call the range and source map, somewhat unintuitively, α and β and thus write α- ... instead.

takes into account the variation of the symplectic structures on the leaves, which is measured by the fundamental class (cf. [23], theorem 1.2.5). This idea allows very precise statements about Morita equivalence of regular Poisson manifolds. For example

Theorem 14 (Xu [23], [24]). *Let P be a regular Poisson manifold with symplectic fibration $\pi\colon P \to Q$. Then P is Morita equivalent to Q with the zero Poisson structure if and only if all the symplectic leaves of P are connected and simply connected and the fundamental class vanishes.*

Theorem 15 (Xu [23], [26]). *Let $\pi\colon P \to M$ be a locally trivial bundle of connected, simply connected symplectic manifolds. Then P is Morita equivalent to M with zero Poisson structure.*

Another interesting result is that Morita equivalent Poisson manifolds have the same zeroth and first cohomology groups [7]. The example of symplectic manifolds shows that no results about higher cohomologies can be expected.

Theorem 13 is an important tool for computing symplectic realizations of Poisson manifolds, by reducing the problem to an (apparently) simpler Morita equivalent manifold (see [23], [26]). For example, Theorem 15 reduces the study of a locally trivial bundle of symplectic manifolds to that of symplectic realizations of the base manifold with zero Poisson structure. Thus in order to classify the complete symplectic realizations of a simply connected, connected, symplectic manifold, we have to classify symplectic realizations of a single point, which is rather easy. Careful bookkeeping shows that every symplectic realization of S is of the form $\mathrm{pr}_S : S \times X \to S$ for a symplectic manifold X, where pr_S is the canonical projection. Another case that can be treated in this way is the orbit space of a Hamiltonian action of a Lie group on a symplectic manifold, and the crossed product of an integrable Poisson manifold with a Lie group acting on it [23], [26].

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