CHAPTER 17

Real and Rational Homotopy Theory

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Est Theorem holds, namely, $H^*(K(R,n);R)$ is a polynomial or exterior algebra on one generator in dimension n, according as n is even or odd. The rational and real theories are then so similar that we can do them simultaneously. Hereafter, we let $\mathbf{R}$ denote R or Q. When $\mathbf{R} = Q$ our theorems apply to $\Delta\mathcal{S}$ and when $\mathbf{R} = R$ to $\Delta\mathcal{T}$ where $\mathcal{T}$ is the category of topological spaces with compactly generated topologies. (See Section 2.) We also view simplicial sets as simplicial spaces with the discrete topology so the real theory also applies to $\Delta\mathcal{S}$. In Section 8, we compare the real and rational theories on $\Delta\mathcal{S}$. They turn out to be substantially different.

Our second direction for extending this theory is to eliminate the nilpotent requirement. We do this by fixing a group π, considering path connected $X \in \Delta\mathcal{T}$ with $\varepsilon : \pi_1(X) \approx \pi$ and localizing X by fibrewise localizing the map X to $B\pi$ defined by ε. The problem is then to make sense out of minimal models in this context. A first approximation would be to take a minimal model for $\tilde{X}$, the universal covering of X. However, this is too crude. For example, one loses the action of π on the higher homotopy groups so that X and $\tilde{X} \times B\pi$ would have the same minimal model. Including a π action on the minimal model for $\tilde{X}$ will give a satisfactory definition of a minimal model for X when π is finite. However, this model will not in general, contain enough information to include all possible k invariants, for example, for adding Z as π_2 to $B\pi$ when $\pi = Z$. A strategy that works for all π is to replace $\mathbf{R}$ by a DG algebra A_0 with a π action which models $\Omega(E\pi)$ (for all local coefficients). One can then define Δ_π, Ω_π and minimal models so that the foundational theorems referred to above hold. (See Section 5.)

The notion of localization being considered here may also be viewed as localizing a category with respect to a set of weak equivalences [14]. For the Quillen–Sullivan rational homotopy theory one considers the category $\Delta\mathcal{S}_{\mathrm{NF}}$ of nilpotent simplicial sets of finite type and as weak equivalences, mappings $f : X \longrightarrow Y$ inducing an isomorphism on rational cohomology. For real homotopy theory one enlarges $\Delta\mathcal{S}_{\mathrm{NF}}$ to $\Delta\mathcal{T}_{\mathrm{NF}}$, the category of nilpotent simplicial spaces of finite type and as weak equivalences, maps which induce an isomorphism on continuous cohomology with coefficients in the reals. In this paper, we in effect consider $\Delta\mathcal{S}_{0,\mathrm{F}}$, the category of connected simplicial sets with base point and finitely generated homotopy groups and as weak equivalences mappings $f : X \longrightarrow Y$ which induce isomorphisms on fundamental groups and on cohomology with local coefficients in Q vector spaces. We also consider $\Delta\mathcal{T}_{0,\mathrm{F}}$, the category of connected simplicial spaces with base point and locally Euclidean homotopy groups and as weak equivalences mappings $f : X \longrightarrow Y$ which induce isomorphisms on fundamental groups and on continuous cohomology with local coefficients in R vector spaces.

2. The categories $\Delta\mathcal{T}_\pi$ and $\mathcal{A}_\pi$

In this section, we introduce the important categories $\Delta\mathcal{T}_\pi$ and $\mathcal{A}_\pi$. We also prove a basic result (Theorem 2.2) relating function spaces and fibrations in these categories.

Recall that for a category $\mathcal{C}$, $A, B \in \mathcal{C}$ means A and B are objects of $\mathcal{C}$ and (A,B) denotes morph(A,B). We denote by $\mathcal{C}_\pi$ the category of $\mathcal{C}$ objects with π actions and π-equivariant maps as morphisms. An object of $\mathcal{C}_\pi$ is a pair (A,ρ) where $A \in \mathcal{C}$ and ρ is a homomorphism of π into (A,A), required to be continuous when the morphisms of with $m_n \to \infty$ as $n \to \infty$, a sequence $\{M_n\}$ of topological π-modules in $\mathcal{T}$, and a sequence of mappings $k^{n+1} : X_{n-1} \to K(M_n, m_{n+1})$ such that $X_n = (X_{n-1})_{k^{n+1}}$ and $\varprojlim X_n$ is homotopy equivalent to X. We say X has a simple Postnikov system if $M_1 = 0$ and $m_n = n$ for all n.

Suppose X is a 0-connected Kan simplicial space with $\pi_1(X) = \pi$ and suppose $\rho : X \longrightarrow B\pi = K(\pi, 1)$ induces an isomorphism on fundamental groups. Let $\tilde{\rho} : \tilde{X} \longrightarrow E(\pi, 0) = E\pi$ be the induced fibration:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\rho}} & E\pi \\ \downarrow & & \downarrow \\ X & \xrightarrow{\rho} & B\pi \end{array}$$

The following is well known:

THEOREM 4.4. *If π is discrete and $(\tilde{X}, \tilde{\rho}) \in (\Delta\mathcal{T}_\pi)_{E\pi}$ is a simplicial set, then $(\tilde{X}, \tilde{\rho})$ has a simple Postnikov system.*

Within the context of our machinery, one can study all simplicial spaces (sets) through the following approach. Let $\Delta\mathcal{T}_\pi$ be the category of pairs (X, ρ) where X is a 0-connected Kan simplicial space (set) with base point and $\rho : X \longrightarrow B\pi$ is a fibration such that $\rho_* : \pi_1(X) \approx \pi_1(B\pi)$. The category $\Delta\mathcal{T}_\pi$ embeds in $(\Delta\mathcal{T}_\pi)_{E\pi}$ by sending (X, ρ) to $(\tilde{X}, \tilde{\rho})$.

DEFINITION 4.5. We say that $(X, \rho) \in \Delta\mathcal{T}_\pi$ has a Postnikov system if $(\tilde{X}, \tilde{\rho})$ has a simple Postnikov system.

THEOREM 4.6. *If $(X, \rho) \in \Delta\mathcal{T}_\pi$ and X is a 0-connected simplicial set (discrete topology) then it has a Postnikov system.*

Suppose $X = B \times_\tau F$ is a twisted Cartesian product of B and F with structural group G, all in $\Delta\mathcal{T}_\pi$. Let $B^{(p)}$ be the p skeleton of B, that is, the smallest simplicial subspace of B containing all B_q, $q \leqslant p$. Filter $C^*(X; M)^\pi$ by

$$F^{p,q} = \{u \in C^{p+q}(X; M)^\pi \,|\, u(B^{(p-1)}_{p+q} \times F_{p+q}) = 0\}.$$

The usual definitions ([15]) then yield the Serre spectral sequence $\{E_r^{p,q}\}$ with its usual relation to $H^*(X; M)$. Let

$$\theta : C^p(B; C^q(F; M)) \longrightarrow C^{p+q}(X; M)$$

be given by

$$\theta(u)(b, f) = u(\partial^q_{p+1} b, \partial^p_0 f)$$