Let \( A \xleftarrow{F} B \) with \( F \dashv U \). Write \( \eta: 1_A \to FU \), \( \epsilon: UF \to 1_B \) for the unit and counit of the adjointness. Then \( T = (T, \eta, \mu) \) is a triple in \( A \), where \( T = FU \), \( \eta: 1_A \to T \), \( \mu: FeU \to T \), and \( T \xrightarrow{T} T \).

We have the category of \( T \)-algebras \( A^T \) as defined by Eilenberg-Moore, \( F^T : A \to A^U \) by \( x \mapsto (XT, Xu) \), \( U^T : A^T \to A \) by \( (x, s) \mapsto x \), and \( F^T \xrightarrow{\mu} U^T \).

\[
\begin{array}{ccc}
A^T & \xrightarrow{\phi} & B \\
U^T \downarrow & & \downarrow \mu \\
A & \xrightarrow{\eta} & U
\end{array}
\]

is defined by \( Y\phi = (YU, YeU) \). The adjoint pair \( F \dashv U \) is tripleable if \( \phi' \xrightarrow{\phi} \phi \) exists such that the unit and counit are isomorphisms \( \eta_A \xrightarrow{\sim} \phi' \phi \), \( \phi' \phi \xrightarrow{\sim} 1_B \). Given \( U \), this property is independent of which left adjoint \( F \) is used, so we also say \( U \) is tripleable in this situation. It seems to be too much to ask for \( \phi' \phi = A^U \), \( \phi' \phi = B \). On the other hand, in category theory, the usual "equivalences" of categories should be replaced by adjoint equivalences.

2. Crude tripleability theorem. If \( B \) has coequalizers and \( U \) preserves and reflects coequalizers, then \( U \) is tripleable.

(It is assumed \( F \dashv U \) exists.)

Proof. \( \phi' \) is the coequalizer \( \xrightarrow{\phi} \).

One way of proving this is by verifying the sequence of set isomorphisms:

- maps \((X, s) \xrightarrow{\phi} Y\phi \)
- \( \to \) maps \( X \xrightarrow{\phi} YU \) such that \( \phi s = F(\psi) \)
- \( \to \) maps \( XT \xrightarrow{\phi} Y \) such that \( \phi T = \psi FG \)
- \( \to \) maps \((X, s) \xrightarrow{\phi} Y \)
If \((X^5) \xrightarrow{\phi} (X^5)\phi^*\phi\) denotes the unit of \(\phi^{-1}\phi\), then \(\phi U = X \eta \tilde{U} kU\).

Now, \(\tilde{X} = \text{coeq}(X^5 U, X^6 F U)\) for if

some \(X^5 U \rightarrow Z\) coequalizes \(X^5 U\) and \(X^6 F U\), then \(X \xrightarrow{X \eta \tilde{U}} Z\) is the unique map such...

But \(kU = \text{coeq}(X^5 U, X^6 U)\) since \(U\) preserves coequalizers. Moreover,

\(\tilde{X}(X^5 U) = \tilde{X} \cdot X \eta \cdot kU = X^5 U \cdot X^5 U \cdot X^6 F U \cdot kU = X^5 U \cdot X^6 F U \cdot kU = X^6 F U \cdot kU\).

Therefore \(X^6 F U \xrightarrow{\eta \phi^* \psi} Y^6 F U\) is an isomorphism, and since \(\tilde{U}\) reflects isomorphism, so is \(\tilde{\phi}\).

The counit \(\phi \phi^* \xrightarrow{\psi} Y\) is defined by its appearance in the diagram below. We proved above that the \(\gamma\)-structure of an algebra is a coequalizer, so if \(U\) is applied to \((X^5 U, X^6 F U, X^6 E U)\), we get a coequalizer diagram in \(A\) (\(X^6 U\) is the \(\gamma\)-structure of the algebra \(X^6\)). But \(U\) reflects coequalizers, so \(Y \xrightarrow{\psi} \text{coeq}(X^5 U, X^6 F U)\). Therefore \(\psi\) is an isomorphism.

**3. Contractible coequalizers.** A diagram \(Y \xrightarrow{d_0} Y_0 \xrightarrow{d_1} Y\) with \(d_0 d = d_1 d\) looks like the 1-skeleton of an augmented simplicial object. (Here degeneracies will be ignored.) A contraction of a simplicial object is a sequence of maps \(h_n : Y_n \rightarrow Y_{n+1}\) such that \(h_n d_i = d_i h_{n-1}\) for \(0 \leq i \leq n\) and \(h_n d_{n+1} = Y_n\). You can also use \(h_n d_0 = Y_n\), \(h_n d_1 = d_{n+1} Y_n\).

We are led to look at diagrams such that \(d_0 d = d_1 d\), \(h_{n+1} d_1 = Y_n\), \(h_0 d_0 = d_1 h_{n-1}\), \(h_0 d_1 = Y_0\). In this case \(d = \text{coeq}(d_0, d_1)\), for if \(d \circ z = z \circ d\) for \(Y_0 \rightarrow Z\) then \(h_{n+1} z : Y \rightarrow Z\) is the unique map such...

Thus we call such a diagram a **contractible coequalizer diagram**.
We call coequalizer data \( Y_1 \xrightarrow{\alpha_1} Y_0 \xrightarrow{\alpha_2} Z \) a contractible coequalizer diagram. We say \( B \) has U-contractible coequalizers if all U-contractible coequalizer data in \( B \) have coequalizers in \( B \); \( U \) preserves U-contractible coequalizers if whenever \( Y_1 \xrightarrow{\alpha_1} Y_0 \xrightarrow{\alpha_2} Y \) is U-contractible and has a coequalizer \( Y_0 \xrightarrow{\gamma} Y \) in \( B \), then the canonical map \( Z \xrightarrow{\gamma} Y_U \) is an isomorphism; \( U \) reflects U-contractible coequalizers if \( Y_1 \xrightarrow{\alpha_1} Y_0 \xrightarrow{\alpha_2} Y \) being mapped into a contractible coequalizer diagram by \( U \) implies that \( Y_1 \xrightarrow{\alpha_1} Y_0 \xrightarrow{\alpha_2} Y \) is a coequalizer diagram in \( B \). (\( Y_i \xrightarrow{\gamma} Y \) will not necessarily be contractible in \( B \).)

4. Precise triple ableness theorem. \( U \) is tripleable \( \iff \) \( B \) has, and \( U \) preserves and reflects, U-contractible coequalizers.

Proof. \( \iff \) is clear. One only has to notice that all coequalizers arising in the proof of the crude theorem were U-contractible.

\( \Rightarrow \): We can assume \( B = A^U \) and prove that \( A^U \) has \( U^U \)-contractible coequalizers. (The (dual) example of comodules over a non-flat coalgebra shows that \( A^U \) need not have all coequalizers. But it follows from a result of Linien's alluded to below that \( A^U \) has all coequalizers if \( A \) is self.) Let \( (X_i, \xi_i) \xrightarrow{\partial_i} (X_0, \xi_0) \) be \( U \)-contractible, i.e. we have the accompanying diagram in \( A \). Let \( X \xrightarrow{\xi} X \) be \( h_{\xi T} \cdot \xi_0 \cdot d \). Then

\[
dT. \xi = \xi_0 \cdot d.
\]

For

\[
dT. \xi = dT. h_{-1} T. \xi_0 \cdot d = (dh_{-1}) T. \xi_0 \cdot d =
\]

For
\[ (h_0, d_0)T \cdot \xi_0 \cdot d_0 = h_0T \cdot \xi_0 \cdot d_0 = h_0T \cdot \xi_1 \cdot d_1 = h_0T \cdot \xi_1 \cdot d_1 = (h_0, d_0)T \cdot \xi_0 \cdot d_0 = \xi_0 \cdot d_0. \]

This shows that \( d : X_0 \to X \) is compatible with \( T \)-structures. Since \( h_1 \cdot d = X \), it follows that \( (X, \xi) \) is a \( T \)-algebra. Also if a different contraction \( h_0', h_1' \) were used, and \( \xi' \) defined as \( h_1' \cdot T \cdot \xi_0 \cdot d_1 \), then

\[ \xi' = \xi, \]

since \( \xi = (h_1, d)T \cdot \xi = h_1T \cdot d \cdot \xi = h_1T \cdot \xi_0 \cdot d_1 \), and \( \xi' = (h_1', d)T \cdot \xi = h_1'T \cdot \xi_0 \cdot d_1 \) also. Thus the \( T \)-structure \( \xi \) is well-defined.

Finally, \( d = \text{coeq} (d_0, d_1) \) for if \( (X_0, \xi_0) \xrightarrow{\delta} (Y, \theta) \) coequalizes \( d_0, d_1 \), then \( (X, \xi) \xrightarrow{\delta} (Y, \theta) \) is the unique \( \star \ystem{(*)} \) The above construction shows that \( UT \) preserves and reflects \( UT \)-contractible coequalizers.

\( \star \) Note that \( h_1 \) is not an algebra map, but \( h_1 \cdot y \) is.

5. Remarks. It should be possible to improve the above theorem (apart from streamlining the exposition). Conditions implying triple-ableness should be found which are easier to verify in practice. For instance, the following \( \star \ystem{(*)} \) true:

\[ U \text{ is tripleable } \iff B \text{ has and } U \text{ preserves } U \text{-contractible coequalizers, and } U \text{ reflects isomorphisms.} \]

It seems to follow without much difficulty, from this, that algebraic or variable categories are tripleable \( \star \ystem{(*)} \) (and Linton can prove tripleable categories are variable).